1 Introduction

The range equation for projectile motion is relatively easy to solve if you assume that the initial vertical position of the projectile equals the final vertical position i.e. the vertical displacement after the projectile has landed is zero. When I saw the derivation (1: p.49-50) for the first time in class, I thought the assumption was rather restrictive which is why I wanted to investigate projectile motion where the final vertical position doesn't equal the initial vertical position. I chose to investigate this setting with the research question:

How does the range of a projectile on an incline line depend on the angle of incline?

2 Theoretical background

2.1 Angles, forces and motion equations

The variables are represented in Figure 1. The range is defined as the distance travelled on the incline line after the projectile has landed. The angle of incline is defined as the angle between the incline line and the horizontal line.

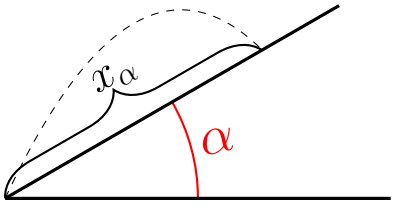


Figure 1: Incline angle α, range x_α and projectile trajectory in dashed line.

To solve for the range along the incline line, new axes are defined based on the angle α. θ, the launch angle, is the angle between the initial velocity vector and horizontal. Therefore the new launch angle is $\theta - \alpha$. Figure 2 illustrates the coordinate system, all relevant angles and the initial velocity and its components.

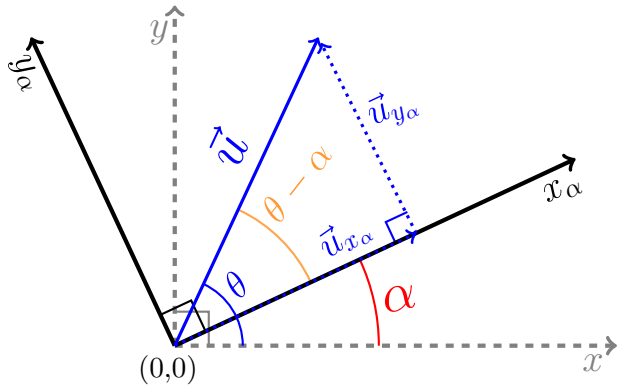


Figure 2: Angles and initial velocity.

Figure 3 shows the forces acting on the projectile while its traveling in the air. The only force acting on the projectile is its weight assuming that all other forces are negligible. In this context, the assumption is that air resistance is negligible (evaluated in Section 6).

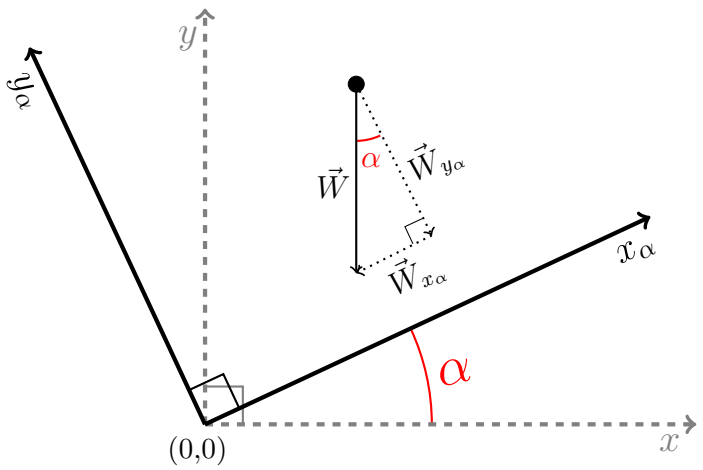


Figure 3: Weight and its components.

To solve for range in projectile motion, the motion is examined in two dimensions separately (1: p.46). The accelerations, based on Figure 3, in the x_α-direction and y_α-direction are:

$$\begin{aligned} F_{x_\alpha} = ma_{x_\alpha} &= W_{x_\alpha} \\ &= W\sin(\alpha) \\ &= -mg\sin(\alpha) \\ \implies a_{x_\alpha} &= -g\sin(\alpha) \end{aligned} \qquad \begin{aligned} F_{y_\alpha} = ma_{y_\alpha} &= W_{y_\alpha} \\ &= W\cos(\alpha) \\ &= -mg\cos(\alpha) \\ \implies a_{y_\alpha} &= -g\cos(\alpha) \end{aligned}$$

From Figure 2 the components of the initial velocity are:

$$u_{x_\alpha} = u\cos(\theta - \alpha) \qquad u_{y_\alpha} = u\sin(\theta - \alpha)$$

The initial position of the projectile is (0,0). Since the acceleration is constant when looking at the direction separately, the Uniformly Accelerated Motion equations (3) can be used.

$$s = ut + \frac{1}{2}at^2$$

Plugging in the accelerations and initial velocities gives the position equations in the x_α- and y_α-direction:

$$x_\alpha = u\cos(\theta - \alpha) \times t - \frac{1}{2}g\sin(\alpha) \times t^2$$
$$y_\alpha = u\sin(\theta - \alpha) \times t - \frac{1}{2}g\cos(\alpha) \times t^2$$

2.2 Deriving the range equation

Solving for the range in the x_α-direction is the same method Tsokos uses to solve for the range in normal projectile motion on horizontal ground (1: p.49-50). The final time is solved from the y_α-equation since when the projectile has landed, its y_α-position is zero. Then this final time is substituted into the x_α-equation. The math is a bit more complicated compared to projectile motion on flat ground as the x_α-equation is a quadratic equation of time and not a linear equation.

$$\begin{aligned} y_\alpha = u\sin(\theta-\alpha)\times t - \frac{1}{2}g\cos(\alpha)\times t^2 &= 0 \\ t\times \quad u\sin(\theta-\alpha) - \frac{1}{2}g\cos(\alpha)\times t \quad &= 0 \\ t_{\text{initial}} &= 0 \\ t_{\text{final}} &= \frac{2u\sin(\theta-\alpha)}{g\cos(\alpha)} \end{aligned}$$

Substituting t_{final} into the x_α-equation

$$\begin{aligned} x_\alpha &= u\cos(\theta-\alpha)\times\frac{2u\sin(\theta-\alpha)}{g\cos(\alpha)} - \frac{1}{2}g\sin(\alpha)\times\left(\frac{2u\sin(\theta-\alpha)}{g\cos(\alpha)}\right)^2 \\ &= \frac{2u^2\cos(\theta-\alpha)\sin(\theta-\alpha)}{g\cos(\alpha)} - \frac{2u^2\sin(\alpha)\sin^2(\theta-\alpha)}{g\cos^2(\alpha)} \\ &= \frac{2u^2\sin(\theta-\alpha)}{g\cos^2(\alpha)}\times[\cos(\alpha)\cos(\theta-\alpha)-\sin(\alpha)\sin(\theta-\alpha)] \end{aligned}$$

Substituting in the cosine rule $\cos A\cos B - \sin A\sin B = \cos(A+B)$

$$\begin{aligned} x_\alpha &= \frac{2u^2\sin(\theta-\alpha)}{g\cos^2(\alpha)}\times\cos(\alpha+(\theta-\alpha)) \\ &= \frac{2u^2\sin(\theta-\alpha)}{g\cos^2(\alpha)}\times\cos(\theta) \end{aligned}$$

The range of the projectile is

$$\boxed{x_\alpha = \frac{2u^2\cos(\theta)\sin(\theta-\alpha)}{g\cos^2(\alpha)}}$$

When $\alpha = 0$, $\sin(\theta-\alpha) = \sin(\theta)$ and $\cos^2(\alpha) = 1$. The equation reverts back to the normal projectile range equation on horizontal ground.

$$x = \frac{2u^2\cos(\theta)\sin(\theta)}{g\times 1} = \frac{u^2\sin(2\theta)}{g}$$

It can be shown that the range equation also applies to decline angles, in which case the angle α would be negative. But to keep the theory and research question concise, I only considered the incline angle.

3 Setting

3.1 Variables

Independent variable
The independent variable is the incline angle. It was measured with an angle meter app on my phone by placing the phone on the incline plank. The angle meter had an instrumental uncertainty of ± 1 .

The Vernier Projectile Launcher gives "repeatability at angles between 0 and 70" (2: p.3). I chose launch angle of 65 to be safe which restricts the incline angle to be less than the launch angle. This gives a range of values of [0,65].

The experiment was repeated every 5 degrees so there were 14 incline angle measures in total.

Dependent variable
The dependent variable is the range. A measuring tape was put on the incline plank and a phone was set up on a tripod to slow-motion record the projectile falling on the plank. The range was recorded later from the videos (Data processing showed more in Section 4).

The Vernier Projectile Launcher gives repeatability "up to a distance of 2.5 m" (2: p.3). The Launcher was set up so that the range would not go over this limit but also be over 2 meters at $\alpha = 0$ so that the fractional uncertainties in the range would be as small as possible.

The experiment was repeated 8 times for every incline angle to reduce effects from random uncertainties. This means there were total of 112 data points.

Controlled variables
Mass of projectile
The same metal ball with mass $m = 21\,\mathrm{g}$ was used every time.

Initial velocity
The initial velocity of the projectile depends on the pressure built up in the chamber. The Launcher allows to set a constant pressure. When the pressure from the hand pump goes over this limit, a valve releases the extra pressure. This is how the pressure was controlled around 100 kPa and thus the initial velocity was controlled.

Launch angle
The launch angle was set at 65 to ensure reliable results with the Launcher. This limited the range of the incline angle values but reliable results were priority.

Wind speed
The experiment was performed indoors so that there would be no effect from wind. Effect of air resistance is evaluated separately in Section 6.

3.2 Setting

List of apparatus

- Vernier Projectile Launcher
- Metal ball
- Hand pump
- Measuring tape
- Plank
- Tripod
- Phone
- Safety goggles

Figure 4: Projectile Launcher and a hand pump.

Figure 5: Full experimental setting.

In Figure 5, the start of the plank is not on the ground because the Launcher Manual says that the launch point is "0.146 m above the base and 0.082 m from the side" (2: p.2) so I tried to get the plank as close to the actual initial position of the metal ball as possible (this is evaluated closer in Section 6).

Method

1. Set up the pressure limit and launch angle of the Launcher
2. Put the ball inside the chamber
3. Start recording the slow-motion video
4. Pump pressure into the chamber until the limit is reached
5. Shoot the projectile (care was taken to ensure that I was never in front of the Launcher when it was pressurized)
6. Stop the slow-motion recording
7. Repeats steps 2 to 6 seven more times for that incline angle

8. Increase incline angle by 5 degrees

9. Repeat steps 2 to 8 thirteen more times

4 Data

The measuring tape used in the experiment had millimiter markings, however, with many of the slow-motion videos it was impossible to record the ball's range with that level of precision. Therefore the range of the ball was taken between centimeter markings illustrated in Figure 6.

The mean range was calculated for each incline angle and the standard deviation was used to get the absolute uncertainty in the mean.

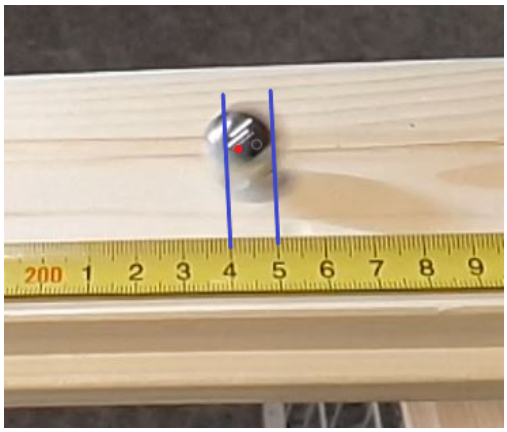


Figure 6: The middle of the ball is between 2.04 m and 2.05 m. This gives the value: $2.045\,\text{m} \pm 0.005\,\text{m}$.

Incline angle	**Range** $x_\alpha/\text{m} \pm 0.005\,\text{m}$							
$\alpha/\ \pm 1$	$x_{\alpha,1}$	$x_{\alpha,2}$	$x_{\alpha,3}$	$x_{\alpha,4}$	$x_{\alpha,5}$	$x_{\alpha,6}$	$x_{\alpha,7}$	$x_{\alpha,8}$
0	2.105	2.135	2.125	2.135	2.125	2.115	2.105	2.115
5	2.045	2.025	2.055	2.045	2.065	2.055	2.055	2.045
10	2.005	1.995	1.985	1.985	2.005	2.005	1.985	1.995
15	1.945	1.945	1.955	1.945	1.955	1.955	1.955	1.945
20	1.915	1.925	1.915	1.915	1.935	1.915	1.925	1.905
25	1.875	1.875	1.885	1.885	1.885	1.895	1.875	1.885
30	1.805	1.825	1.815	1.805	1.825	1.815	1.815	1.825
35	1.785	1.805	1.805	1.805	1.805	1.795	1.795	1.775
40	1.775	1.755	1.765	1.765	1.765	1.755	1.775	1.775
45	1.705	1.705	1.715	1.705	1.715	1.705	1.695	1.695
50	1.615	1.605	1.615	1.605	1.605	1.615	1.605	1.625
55	1.445	1.465	1.455	1.455	1.455	1.445	1.445	1.425
60	1.165	1.165	1.175	1.175	1.175	1.175	1.165	1.175
65	0.775	0.765	0.785	0.775	0.775	0.775	0.785	0.765

Table 1: Raw data.

$$\overline{x}_\alpha = \frac{\sum_{n=1}^{8} (x_{\alpha,n})}{8} \qquad \Delta\overline{x}_\alpha = \sqrt{\frac{1}{8-1} \times \sum_{n=1}^{8} (x_{\alpha,n} - \overline{x}_\alpha)^2}$$

(Standard deviation formula)

Incline angle	**Mean range**	**Uncertainty in mean range**
$\alpha / \ \pm 1$	$\overline{x}_\alpha$/m	$\Delta\overline{x}_\alpha$/m
0	2.12	0.01
5	2.05	0.01
10	1.995	0.009
15	1.950	0.005
20	1.919	0.009
25	1.882	0.007
30	1.816	0.008
35	1.80	0.01
40	1.766	0.008
45	1.705	0.008
50	1.611	0.007
55	1.45	0.01
60	1.171	0.005
65	0.775	0.008

Table 2: Final data set. Absolute uncertainties rounded to one significant figure and mean ranges rounded to the same decimal place as corresponding uncertainty.

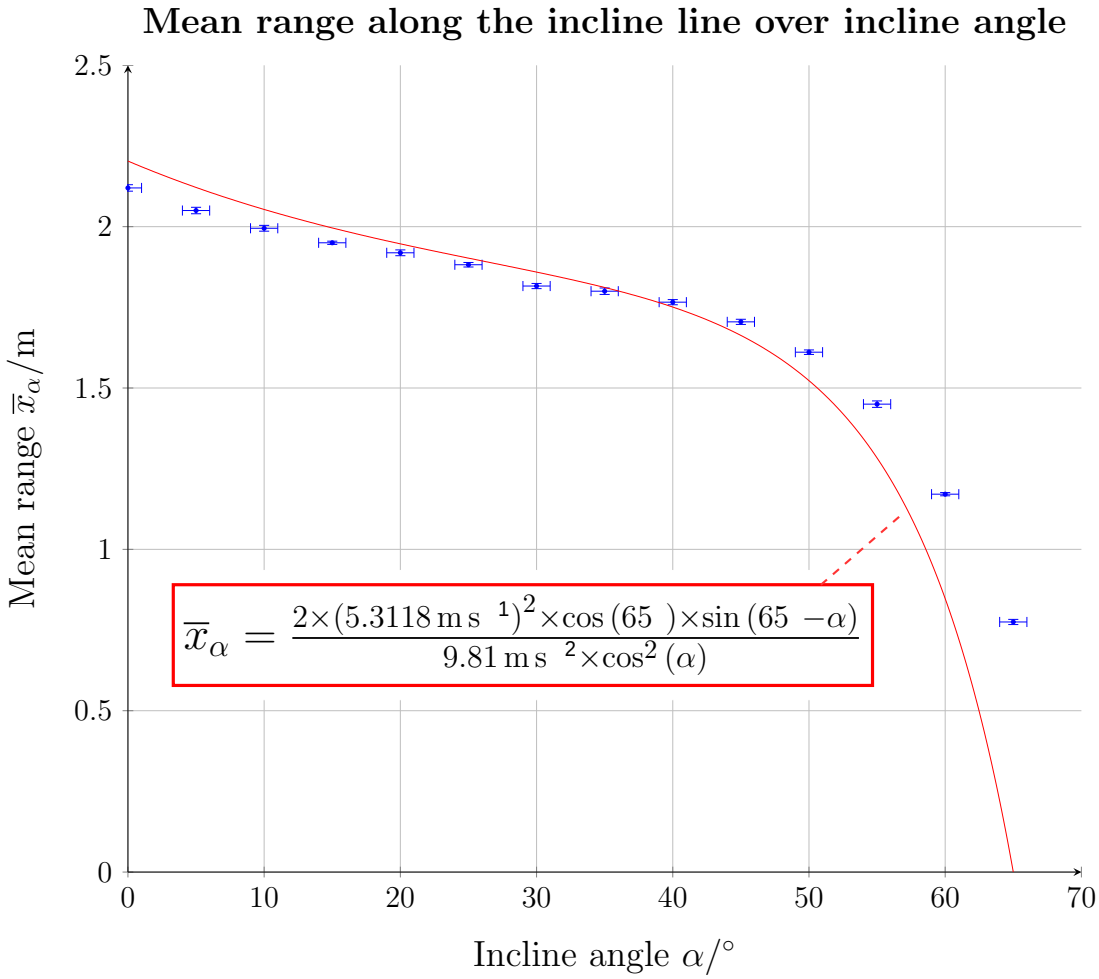
Linearizing the data could show systematic uncertainties better in the x_α-axis. However, the relationship can't be linearized which means that the data has to be used as is.

The only unknown variable in the range equation is the initial velocity u. The Projectile Launcher allows to measure the initial velocity by measuring the time it takes for the projectile to travel a very short distance in the chamber (2: p.3). The initial velocity was measured five times:

$$u = \frac{(5.284 + 5.321 + 5.328 + 5.309 + 5.317)\,\mathrm{m\,s^{-1}}}{5} = 5.3118\,\mathrm{m\,s^{-1}}$$

Therefore, based on the theoretical background in Section 3, the data points should fall on the line:

$$x_\alpha = \frac{2 \times (5.3118\,\mathrm{m\,s^{-1}})^2 \times \cos(65^\circ) \times \sin(65^\circ - \alpha)}{9.81\,\mathrm{m\,s^{-2}} \times \cos^2(\alpha)}$$



5 Analysis

5.1 Random uncertainties

The absolute uncertainties in Table 2 are small compared to the range values. The percentage uncertainty can be calculated with:

$$\frac{\Delta\overline{x}_\alpha}{\overline{x}_\alpha} \times 100$$

Incline angle α	0	5	10	15	20	25	30	35	40	45	50	55	60	65
Mean range %-unc.	0.5	0.5	0.5	0.3	0.5	0.4	0.4	0.6	0.5	0.5	0.4	0.7	0.4	1

Table 3: Percentage uncertainty of mean range values.

Table 3 shows that the percentage uncertainties are less than one. This means that the experimental setting was precise and random uncertainties didn't have a big effect on the results.

5.2 Conclusion

The data points follow some of the same characteristics as the model-line. The range in the data points is constantly decreasing as the incline angle increases. The rate of range decrease is largest at the extreme incline angle values $\alpha = 0$ and $\alpha = 65$ and the range in the data points seems to approach zero between $\alpha = 65$ and $\alpha = 70$ just like the model does.

However, the data points do deviate from the model significantly at times. In fact most of the points (11/14) don't touch the line with the error bars. The points less than 35 are lower than the model and the deviation seems to increase as the angle approaches 0 . The data points more than 35 are higher than the model and the deviation seems to increase as the angle approaches 65 . The percentage difference between the data points and the model can be calculated with:

$$\frac{\overline{x}_{\alpha,\text{experimental}} - x_{\alpha,\text{theoretical}}}{x_{\alpha,\text{theoretical}}} \times 100 \tag{1}$$

Incline angle α	0	5	10	15	20	25	30	35	40	45	50	55	60	65
Mean range %-diff.	4	4	3	3	2	1	3	1	0.6	2	6	10	40	*

Table 4: Percentage difference to the theoretical model.

From Table 4 it's clear that the difference gets exceptionally large when the incline angle gets closer to the launch angle. There isn't value for $\alpha = 65$ because the theoretical range is $0\,\text{m}$ so there would be division by zero in Equation 1. The model makes physical sense since if the incline angle equals, or is more then the launch angle, the projectile obviously shouldn't go anywhere. But in the experiment the range at this angle was $x_\alpha = 0.775\,\text{m}$.

Despite the similar characteristics of the data points and the model-line the deviation is large at times and since this deviation is not constant it warrants a closer look. The reason for it and the trend it creates is explained next.

6 Evaluation

6.1 Initial position

6.1.1 Small angles

Most of the deviation from the model can be explained with the initial position of the projectile. As said in Section 3, the projectile is "0.416 m above the base and 0.082 m from the side" and I tried to compensate for this by raising the plank off the ground (Figure 5). However the horizontal difference meant that, with small incline angles, the measured ranges were smaller than what they were in reality. Figure 7 shows how the initial position affected the range measurement.

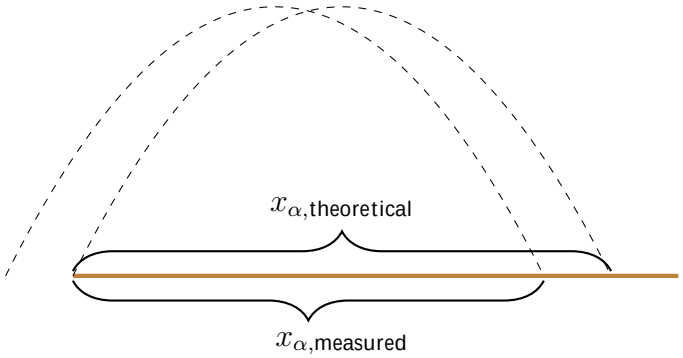


Figure 7: The measured range is smaller compared to if the initial position was right at the start of the plank.

The difference between the measured range and the model range at $\alpha = 0$ is:

$$\frac{2 \times (5.3188\,\text{m s}^{-1})^2 \times \cos(65^\circ) \times \sin(65^\circ - 0^\circ)}{(9.81\,\text{m s}^{-2}) \times \cos^2(0^\circ)} - 2.12\,\text{m} \approx 0.089\,\text{m}$$

This value is very close to the actual distance the projectile was left of the edge of the Launcher (0.082 m). This implies that the initial position accounts for vast majority of the deviation from the model.

6.1.2 Large angles

Already during the experiment it became clear that the initial position would have a large effect when the incline angle approached the launch angle. The effect reverses from small angles and the measured range values are larger than they are supposed to be. Figure 8 shows how the initial position affects the range with a large incline angle. Figure 9 shows how the setting in real life looked when the incline angle was $\alpha = 65^\circ$: the projectile travels almost parallel to the plank until finally hitting it.

6.2 Other factors

The derivation of the range equation in Section 2 assumes that air resistance is negligible so that the motion equations would be simpler. However, air resistance has a real effect in